***15.** Prove that if X and Z are cobordant in Y, then for every compact manifold C in Y with dimension complementary to X and Z, $I_2(X, C) = I_2(Z, C)$. [HINT: Let f be the restriction to W of the projection map $Y \times I \longrightarrow Y$, and use the Boundary Theorem.]

16. Prove that $\deg_2(f)$ is well defined by direct application of the Boundary Theorem. [HINT: If $y_0, y_1 \in Y$, alter f homotopically to get $f \pitchfork \{y_0, y_1\}$. Now let $c: I \longrightarrow Y$ be a curve with $c(0) = y_0$, $c(1) = y_1$, and define $F: X \times I \longrightarrow Y \times Y$ by $f \times c$. Examine ∂F.]

17. Derive the Nonretraction Theorem of Section 2 from the Boundary Theorem.

18. Suppose that Z is a compact submanifold of Y with $\dim Z = \frac{1}{2} \dim Y$. Prove that if Z is globally definable by independent functions, then $I_2(Z, Z) = 0$. [HINT: By Exercise 20, Section 3, $N(Z; Y) = Z \times \mathbf{R}^k$. Certainly $I_2(Z, Z) = 0$ in $Z \times \mathbf{R}^k$, since $Z \times \{a\} \cap Z$ is empty. Now use the Tubular Neighborhood Theorem, Exercise 16, Section 3.]

19. Show that the central circle X in the open Möbius band has mod 2 intersection number $I_2(X, X) = 1$. [HINT: Show that when the ends of the strip in Figure 2-18 are glued together with a twist, X' becomes a manifold that is a deformation of X.]

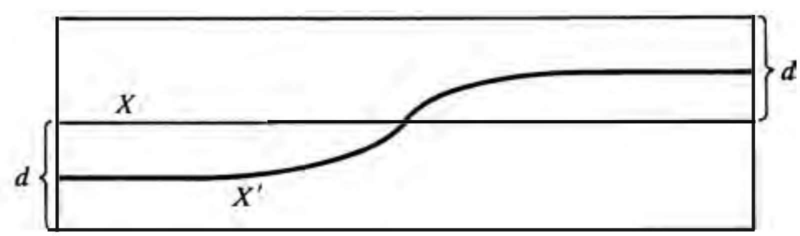

Figure 2-18

Corollary to Exercises 19 and 20. The central circle in the Möbius band is not definable by an independent function.

§5 Winding Numbers and the Jordan-Brouwer Separation Theorem

The Classical Jordan Curve Theorem says that every simple closed curve in $\mathbf{R}^2$ divides the plane into two pieces, the "inside" and "outside" of the curve. Lest the theorem appear too obvious, try your intuition on the example shown in Figure 2-19.

This section is a self-guided expedition with gun and camera into the wilds of such jungles, and in n dimensions, too! Begin with a compact, con-

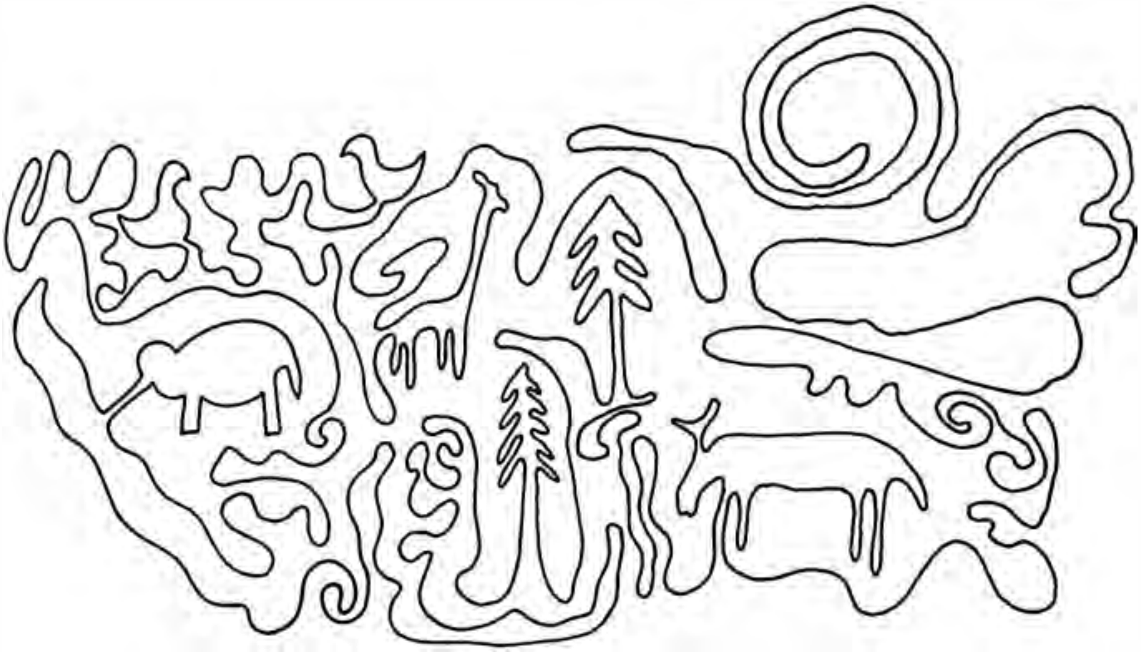

Figure 2-19

nected manifold x and a smooth map $f: X \longrightarrow \mathbf{R}^n$. Suppose that $\dim X = n - 1$, so that, in particular, f might be the inclusion map of a hypersurface into $\mathbf{R}^n$. (In general, a *hypersurface* in a manifold is a submanifold of codimension one.) We wish to study how f wraps X around in $\mathbf{R}^n$, so take any point z of $\mathbf{R}^n$ not lying in the image $f(X)$. To see how $f(x)$ winds around z, we inquire how often the unit vector

$$u(x) = \frac{f(x) - z}{|f(x) - z|},$$

which indicates the direction from z to $f(x)$, points in a given direction. From Intersection Theory, we know that $u: X \longrightarrow S^{n-1}$ hits almost every direction vector the same number of times mod 2, namely, $\deg_2(u)$ times. So seize this invariant and define the *mod 2 winding number* of f around z to be $W_2(f, z) = \deg_2(u)$. (See Figure 2-20.)

In a moment you will use this notion (i.e., mod 2 winding number) to establish a generalized version of the Jordan curve theorem, but first some exercises will develop a preliminary theorem. The proof introduces a beauti-

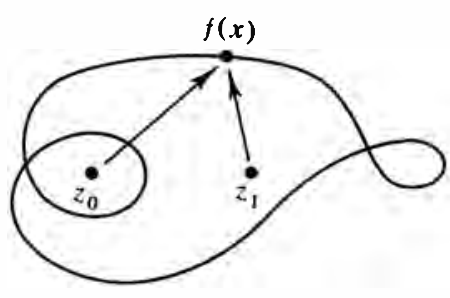

Figure 2-20

fully simple technique that appears repeatedly in later sections. Hints are provided at the end of the section, but it should be fun to fit the proof together yourself.

Theorem. Suppose that X is the boundary of D, a compact manifold with boundary, and let $F: D \longrightarrow \mathbf{R}^n$ be a smooth map extending f; that is, $\partial F = f$. Suppose that z is a regular value of F that does not belong to the image of f. Then $F^{-1}(z)$ is a finite set, and $W_2(f, z) = \# F^{-1}(z) \bmod 2$. That is, f winds X around z as often as F hits z, mod 2. (See Figure 2-21.)

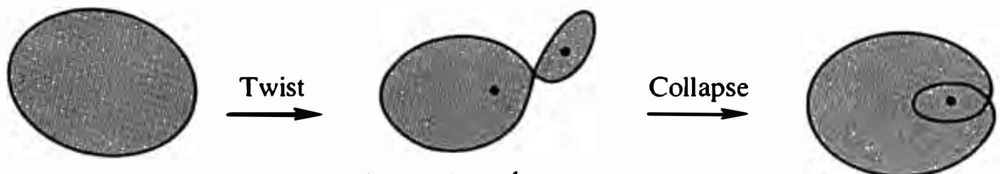

Figure 2-21

Here are some exercises to help you construct a proof:

1. Show that if F does not hit z, then $W_2(f, z) = 0$.

2. Suppose that $F^{-1}(z) = \{y_1, \ldots, y_l\}$, and around each point y_i let B_i be a ball, (That is, B_i is the image of a ball in $\mathbf{R}^n$ via some local parametrization of D.) Demand that the balls be disjoint from one another and from $X = \partial D$. Let $f_i: \partial B_i \longrightarrow \mathbf{R}^n$ be the restriction of F, and prove that

$$W_2(f, z) = W_2(f_1, z) + \cdots + W_2(f_l, z) \mod 2.$$

(See Figure 2-22.)

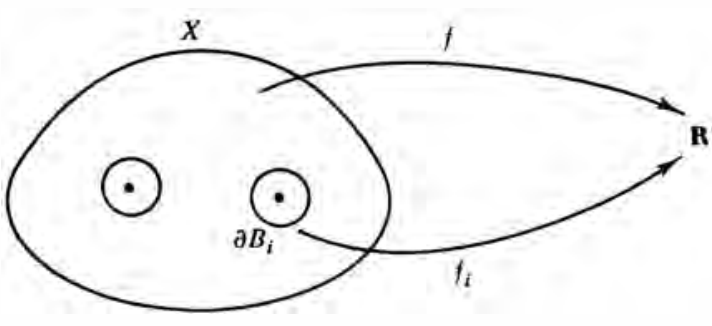

Figure 2-22

3. Use the regularity of z to choose the balls B_i so that $W_2(f_i, z) = 1$, and thus prove the theorem.

Now assume that X actually is a compact, connected hypersurface in $\mathbf{R}^n$. If X really does separate $\mathbf{R}^n$ into an inside and an outside, then it should be the boundary of a compact n-dimensional manifold with boundary—namely, its inside. In this case, the preceding theorem tells us that if $z \in \mathbf{R}^n$ is any point not on X, then $W_2(X, z)$ must be 1 or 0, depending on whether z lies inside or outside of X. [Here $W_2(X, z)$ is used for the winding number of the inclusion map of X around z.] The next exercises help you reverse this reasoning to prove the Separation Theorem.

4. Let $z \in \mathbf{R}^n - X$. Prove that if x is any point of X and U any neighborhood of x in $\mathbf{R}^n$, then there exists a point of U that may be joined to z by a curve not intersecting X (Figure 2-23).

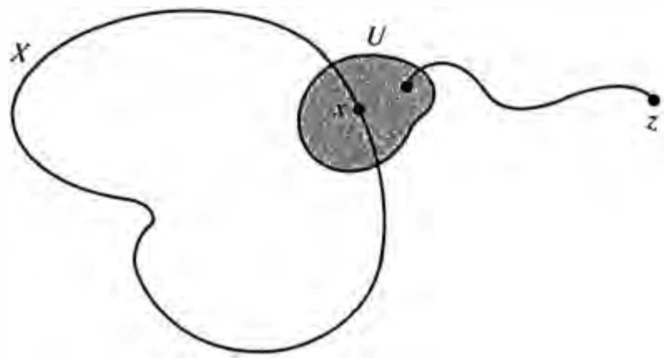

Figure 2-23

5. Show that $\mathbf{R}^n - X$ has, at most, two connected components.

6. Show that if z_0 and z_1 belong to the same connected component of $\mathbf{R}^n - X$, then $W_2(X, z_0) = W_2(X, z_1)$.

7. Given a point $z \in \mathbf{R}^n - X$ and a direction vector $\vec{v} \in S^{n-1}$, consider the ray r emanating from z in the direction of $\vec{v}$,

$$r = \{z + t\vec{v} : t \geq 0\}.$$

Check that the ray r is transversal to X if and only if $\vec{v}$ is a regular value of the direction map $u: X \longrightarrow S^{n-1}$. In particular, almost every ray from z intersects X transversally.

8. Suppose that r is a ray emanating from z_0 that intersects X transversally in a nonempty (necessarily finite) set. Suppose that z_1 is any other point on r (but not on X), and let l be the number of times r intersects X between z_0 and z_1. Verify that $W_2(X, z_0) = W_2(X, z_1) + l \mod 2$. (See Figure 2-24.)

9. Conclude that $\mathbf{R}^n - X$ has precisely two components,

$$D_0 = \{z : W_2(X, z) = 0\} \quad \text{and} \quad D_1 = \{z : W_2(X, z) = 1\}.$$

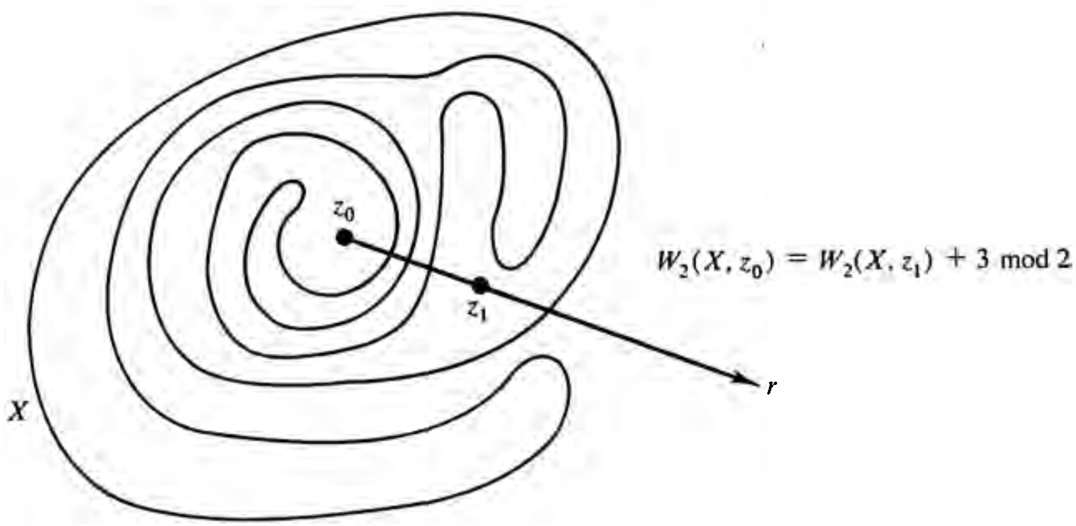

Figure 2-24

10. Show that if z is very large, then $W_2(X, z) = 0$.

11. Combine these observations to prove

The Jordan-Brouwer Separation Theorem. The complement of the compact, connected hypersurface X in $\mathbf{R}^n$ consists of two connected open sets, the "outside" D_0 and the "inside" D_1. Moreover, $\bar{D}_1$ is a compact manifold with boundary $\partial \bar{D}_1 = X$.

Note that we have actually derived a simple procedure for determining whether a given point z lies inside or outside of X.

12. Given $z \in \mathbf{R}^n - X$, let r be any ray emanating from z that is transversal to X. Show that z is inside X if and only if r intersects X in an odd number of points (Figure 2-25).

Hints (listed by exercise number)

1. If u extends to D, then $\deg_2(u) = 0$.
2. Use 1, replacing D by

$$D' = D - \bigcup_{i=1}^{l} \operatorname{Int}(B_i).$$

3. If f_i carries ∂B_i diffeomorphically onto a small sphere centered at z, then $u_i : \partial B_i \longrightarrow S^{n-1}$ is bijective. But f is a local diffeomorphism at y_i, so you can choose such B_i.
4. Show that the points $x \in X$ for which the statement is true constitute a nonempty, open, and closed set. (Closed: easy. Open: use

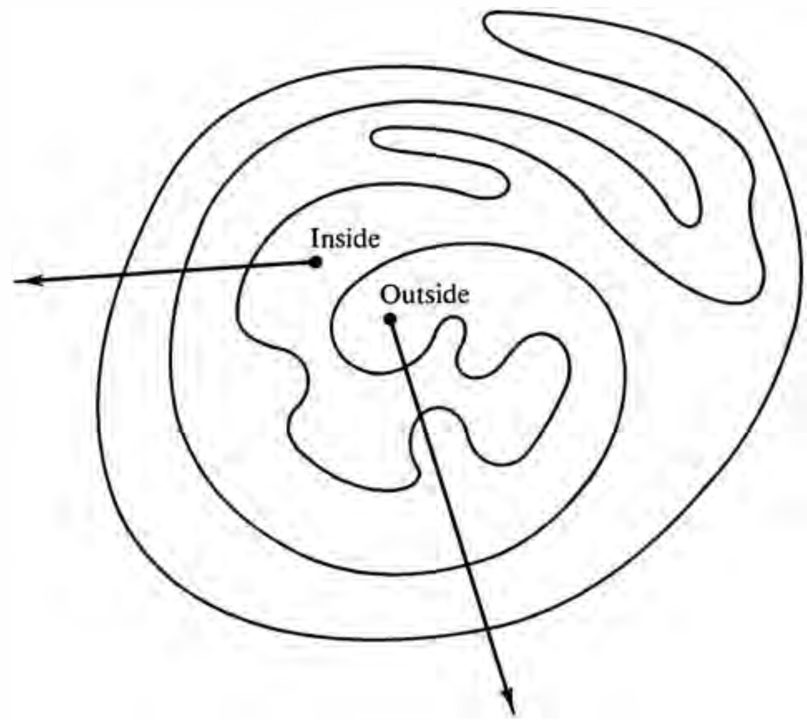

Figure 2-25

coordinates making X look locally like a piece of $\mathbf{R}^{n-1}$ in $\mathbf{R}^n$. Nonempty: consider the straight line joining z to the closest point in X.)

5. Let B be a small ball such that $B - X$ has two components, and fix points z_0 and z_1 in opposite components. Every point of $\mathbf{R}^n - X$ may be joined to either z_0 or z_1 by a curve not intersecting X. (We have not yet excluded the possibility that z_0 and z_1 may be so joined!)
6. If z_t is a curve in $\mathbf{R}^n - X$, then the homotopy

$$u_t(x) = \frac{x - z_t}{|x - z_t|}$$

is defined for all t. Thus u_0 and u_1 have the same mod 2 degree.

7. Either compute directly or use Exercise 7, Chapter 1, Section 5. Note that if $g: \mathbf{R}^n - \{z\} \longrightarrow S^{n-1}$ is $g(y) = y - z/|y - z|$, then $u: X \longrightarrow S^{n-1}$ is g composed with the inclusion map of X. So by the exercise cited, $u \pitchfork \{\vec{v}\}$ if and only if $X \pitchfork g^{-1}\{\vec{v}\}$.
8. By Exercise 7, $\vec{v}$ is a regular value for both u_0 and u_1. But

$$\# u_0^{-1}(\vec{v}) = \# u_1^{-1}(\vec{v}) + l.$$

9. Exercise 8 implies that both D_0 and D_1 are nonempty. Now apply Exercises 5 and 6.
10. Since X is compact, when $|z|$ is large the image $u(X)$ on S^{n-1} lies in a small neighborhood of $z/|z|$.

11. By Exercise 10, D_1 is compact, and $\bar{D}_1 = D_1 \cup X$. Produce a local parametrization of $\bar{D}_1$ around a point $x \in X$ as follows: Let $\psi : B \longrightarrow \mathbf{R}^n$ map a ball B around 0 in $\mathbf{R}^n$ diffeomorphically onto a neighborhood of x in $\mathbf{R}^n$, carrying $B \cap \mathbf{R}^{n-1}$ onto $X \cap \psi(B)$. Use Exercises 4 and 6 to prove either that $\psi(B \cap H^n) \subset D_1$ and $\psi(B \cap -H^n) \subset D_0$ or the reverse. In either case, ψ restricts to a local paramterization of $\bar{D}_1$ around x.

12. Combine Exercise 8 with Exercise 10.

§6 The Borsuk-Ulam Theorem

We shall use our winding number apparatus to prove another famous theorem from topology, the Borsuk-Ulam theorem. One form of it is

Borsuk-Ulam Theorem. Let $f: S^k \longrightarrow \mathbf{R}^{k+1}$ be a smooth map whose image does not contain the origin, and suppose that f satisfies the symmetry condition

$$f(-x) = -f(x) \qquad \text{for all } x \in S^k,$$

Then $W_2(f, 0) = 1$.

Informally, any map that is symmetric around the origin must wind around it an odd number of times.

Proof. Proceed by induction on k. For $k = 1$, see Exercise 2.

Now assume the theorem true for $k - 1$, and let $f: S^k \longrightarrow R^{k+1} - \{0\}$ be symmetric. Consider S^{k-1} to be the equator of S^k, embedded by $(x_1, \ldots, x_k) \longrightarrow (x_1, \ldots, x_k, 0)$. The idea of the proof is rather like Exercise 12 in the previous section. We will compute $W_2(f, 0)$ by counting how often f intersects a line l in $\mathbf{R}^{k+1}$. By choosing l disjoint from the image of the equator, we can use the inductive hypothesis to show that the equator winds around l an odd number of times. Finally, it is easy to calculate the intersection of f with l once we know the behavior of f on the equator.

Denote the restriction of f to the equator S^{k-1} by g. In choosing a suitable line l, use Sard to select a unit vector $\vec{a}$ that is a regular value for both maps

$$\frac{g}{|g|}: S^{k-1} \longrightarrow S^k \quad \text{and} \quad \frac{f}{|f|}: S^k \longrightarrow S^k.$$

From symmetry, it is clear that $-\vec{a}$ is also a regular value for both maps. By dimensional comparison, regularity for $g/|g|$ simply means that $g/|g|$ never hits $\vec{a}$ or $-\vec{a}$; consequently, g never intersects the line $l = \mathbf{R} \cdot \vec{a}$. We let you check that regularity for $f/|f|$ is equivalent to the condition $f \pitchfork l$. (See Exercise 7, Chapter 1, Section 5.)